Lecture 17.

- Cauchy's Integral Formula – Baby version. Statement + pf from Lecture 16 notes. (Lemma 1 proved last time.)

Thm 1. Let f be analytic in $B(a,R)$. Then, $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ w/ R.O.C. $\geq R$. Moreover, f has ∞ many $\mathbb{C}$-der. in $B(a,R)$ and $a_n = \frac{1}{n!} f^{(n)}(a)$.

Pf. WLOG $a=0$. We prove that $\forall r<R$ $f(z) = \sum_{n=0}^{\infty} a_n z^n$ w/ abs. + unif. conv. in $\overline{B(a,r)}$ $\Rightarrow$ conclusion by previous result on p.s.
By CIF-Baby, we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\zeta)\,d\zeta}{\zeta - z}, \quad z \in \overline{B(0,r)}$$

where $\gamma(s) = \rho e^{is}$, $s \in [0, 2\pi]$, and $r < \rho < R$.

For $|\zeta| = \rho$ and fixed $|z| \le r$, we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\,\frac{1}{1 - z/\zeta} = \frac{1}{\zeta}\sum_{n=0}^{\infty}\left(\frac{z}{\zeta}\right)^n$$

w/ abs. + unif. conv. for $\zeta \in \gamma$,

since $\displaystyle\sum_{n=0}^{\infty}\left|\frac{z}{\zeta}\right|^n \le \sum\left(\frac{r}{\rho}\right)^n < \infty.$

By UG analysis, we may interchange $\int$ and Σ $\Rightarrow$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = \sum_{n=0}^{\infty} z^n \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta$$

$$= \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}}\,d\zeta.$$

The conv. is again abs. + unif. since

$$|a_n| \le \max_{\zeta\in\gamma}|f(\zeta)| \cdot \frac{1}{\rho^{n+1}}\,\frac{1}{2\pi}\int_\gamma |d\zeta|$$

$$= M \cdot \frac{1}{\rho^n} \Rightarrow \sum_{n=0}^{\infty}|a_n|\cdot|z^n| \le \sum_{n=0}^{\infty}\left(\frac{r}{\rho}\right)^n < \infty.$$

Thus, for any $r<R$, $f(z)=\sum_{n=0}^{\infty} a_n z^n$ w/ abs. + unif. conv. on $\overline{B(0,r)}$. It follows that f has ∞ many $\mathbb{C}$-der. given by differentiating the series termwise, in particular, $a_n = \frac{1}{n!} f^{(n)}(0)$. This completes proof. ☒

We note that in the pf, we showed that if f is analytic in $B(a,R)$ then $\forall\, r<R$

$$a_n = \frac{1}{n!} f^{(n)}(a) = \frac{1}{2\pi i} \int_{\gamma_{r}} \frac{f(\zeta)}{(\zeta-a)^{n+1}}\, d\zeta, \quad |z-a|=r$$

which implies

$$\frac{1}{n!}\left|f^{(n)}(a)\right| \le \underset{\max\limits_{\zeta\in\gamma_r} |f(\zeta)|}{M_r} \cdot \frac{1}{r^n}.$$

$\Rightarrow$

<u>Cauchy's Estimate</u> If f is analytic and $|f|\le M$ in $B(a,R)$, then

$$\left|f^{(n)}(a)\right| \le \frac{n!\,M}{R^n}.$$

Another important consequence of Thm 1 is

Cor 1. If f is analytic in $B(a,r)$, then

(i) f has a primitive F in $B(a,r)$

(ii) (Cauchy's Thm - Baby) If γ is a closed curve in $B(a,r)$, then

$$\int_\gamma f\,dz = 0.$$

Pf. (i) Let $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$, and take $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(z-a)^{n+1}$, which is easily seen to have same R.O.C. ($\geq R$) and satisfies $F' = f$.

(ii) Follows from (i) and FTC. ∎